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Optimal lower bounds on the electric-field concentration in composite media

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Composites made from two linear-isotropic-dielectric materials are considered. It is assumed that only the volume fraction and the two-point correlation function of each dielectric material are known. Lower bounds on all rth moments of the electric-field intensity inside each phase are obtained for $r \geqslant 2$. A lower bound on the maximum field intensity inside the composite is also obtained. The bounds are given in terms of the one- and two-point statistics of the microgeometry. All of these bounds are shown to be the best possible as they are attained by the electric-field associated with a suitably constructed space-filling confocal-ellipsoid assemblage. The bounds provide a new opportunity for the assessment of local field behavior in terms of a statistical description of the microstructure. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1777808]

I. INTRODUCTION

The study of failure initiation in dielectric composites requires one to assess the magnitude of the local electric-field arising from macroscopic potential gradients. Macroscopic quantities sensitive to the local field behavior include higher-order moments of the electric-field inside the composite. In this work, we focus on two–phase dielectric composites and develop optimal lower bounds on the higher moments of the electric-field that depend on statistics of the microgeometry gathered from image analysis.[1]

The composite is contained inside a cube Q, and no constraints are placed upon the arrangement of the two materials inside Q. The subsets of Q, occupied by materials one and two, are denoted by Q_1 and Q_2, respectively. The indicator function of material one is denoted by χ_1 and takes the value one inside Q_1 and zero outside. The indicator function of material two is given by χ_2 and $\chi_2 = 1 - \chi_1$. It is supposed that Q is the period cell for an infinite periodic medium. The one-point and two-point correlation functions are given by

$$S_1^1 = \frac{1}{|Q|}\int_Q \chi_1(\mathbf{x})d\mathbf{x}$$

and

$$S_1^2(\mathbf{t}) = \frac{1}{|Q|}\int_Q \chi_1(\mathbf{x})\chi_1(\mathbf{x}+\mathbf{t})d\mathbf{x}, \tag{1.1}$$

where $\mathbf{t}$ is any vector and $|Q|$ is the volume of Q. The one-point correlation S_1^1 gives the volume fraction of material one. The two-point correlation $S_1^2(\mathbf{t})$ gives the probability that a rod of length and orientation specified by $\mathbf{t}$ has both ends in material one when it is translated and dropped inside the periodic medium. Image-processing techniques have recently been developed in Ref. 1 to determine the one-point, two-point, and three-point correlation functions from images of composite microstructure.

The electric and displacement fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{D}(\mathbf{x})$ inside the two-phase dielectric satisfy $\mathbf{E}(\mathbf{x}) = \nabla\phi(\mathbf{x})$ and $\mathbf{D}(\mathbf{x}) = \epsilon(\mathbf{x})\mathbf{E}(\mathbf{x})$. Here, $-\phi$ is the electric potential and the dielectric constant $\epsilon(\mathbf{x})$ takes the two values ϵ_1 and ϵ_2, with $\epsilon_1 > \epsilon_2$, and

$$\begin{aligned} &\Delta\phi = 0, \text{ in phase } 1,\\ &\Delta\phi = 0, \text{ in phase } 2. \end{aligned} \tag{1.2}$$

It is assumed that there is perfect contact between the dielectrics so the electric potential and normal component of the displacement are continuous across the two-phase interface, i.e.,

$$\begin{aligned} &\phi_1 = \phi_2,\\ &\mathbf{D}_1\cdot\mathbf{n} = \mathbf{D}_2\cdot\mathbf{n}. \end{aligned} \tag{1.3}$$

Here, $\mathbf{n}$ is the unit normal to the interface pointing into material two, and the subscripts indicate the side of the interface that the fields are evaluated on. For a prescribed constant electric-field $\overline{\mathbf{E}}$, the average electric-field $\langle\mathbf{E}\rangle$ satisfies $\langle\mathbf{E}\rangle = \overline{\mathbf{E}}$ and $\phi(\mathbf{x}) - \overline{\mathbf{E}}\cdot\mathbf{x}$ is periodic on Q. The effective dielectric tensor is defined by

$$\langle\mathbf{D}\rangle = \mathcal{E}^e\overline{\mathbf{E}}. \tag{1.4}$$

In this work, we consider the moments of the electric-field intensity inside each phase given by

$$\langle\chi_1|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r}$$

and

$$\langle\chi_2|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} \tag{1.5}$$

for $2 \leqslant r < \infty$. Here, $\langle\cdot\rangle$ indicates the volume average of a quantity over the cube Q. We also consider the L^∞ norms given by

$$\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_1)} = \lim_{r\to\infty}\langle\chi_1|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r},$$

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$$\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)} = \lim_{r\to\infty}\langle\chi_2|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r},$$

$$\|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)} = \lim_{r\to\infty}\langle|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r}. \tag{1.6}$$

In Sec. III, we present explicit optimal lower bounds on the moments in Eq. (1.5) and L^∞ norms in Eq. (1.6) that are given in terms of S_1^1 and S_1^2. In this, work the minimizing configurations are shown to be given by suitably constructed confocal-ellipsoid assemblages.[2,3] These configurations include the Hashin–Shtrikman[4] coated sphere assemblage as a special case. The bounding technique presented in Secs. II and III also applies when loss becomes significant in the dielectrics, i.e., for complex values of ϵ_1 and ϵ_2. This issue is taken up in Sec. VII where explicit bounds are given for statistically isotropic composites.

The optimal lower bounds for the higher moments of the electric-fields can be used to assess the effective higher-order response of weakly nonlinear composite media. This is due to the fact that the effective higher-order nonlinear response for weakly nonlinear dielectric media can be approximated by suitable higher moments of the linear electric-field.[5–8]

Previous investigations have provided upper and lower bounds on the second moments of the electric-field in composite media.[9–14] Higher-order moments of the electric-field have been calculated numerically for the two-dimensional dispersions of disk, needle, and square-shaped inclusions[15] as well as the density of states for the Hashin coated cylinder assemblage.[16] For the multiphase nonlinear power law dielectric composites, optimal lower bounds on the moments of the electric-field are found when the degree of the moment matches the power of the nonlinearity.[17] For completeness, we list recent work done in the context of two-phase linear elasticity. Here, optimal inclusion shapes are sought that minimize the maximum eigenvalue of the local stress for a given applied stress. The work presented in Ref. 18 provides an optimal lower bound on the supremum of the maximum principle stress for a single, simply connected stiff inclusion in an infinite matrix subject to a remote stress at infinity. The optimal shapes are given by ellipsoids. The work presented in Ref. 19 provides an optimal lower bound on the supremum of the maximum principle stress for two-dimensional periodic composites consisting of a single, simply connected stiff inclusion in the period cell. The bound is given in terms of the area fraction of the included phase, and for an explicit range of prescribed average stress, the optimal inclusions are given by the Vigdergauz shapes.[20]

II. LOWER BOUNDS ON THE ELECTRIC-FIELD INTENSITY IN ANISOTROPIC COMPOSITES AND SUFFICIENT CONDITIONS FOR OPTIMALITY

In this section, we establish lower bounds on the L^∞ norm of the electric-field inside each material. Sufficient conditions on the electric-field are identified, which guarantee that lower bound is attained. These conditions are used to establish the optimality of the bounds presented in Sec. III.

For $0<\theta_1<1$, we suppose that the volume fraction of material one S_1^1 is fixed and given by θ_1. The volume fraction of material two is given by $\theta_2=1-\theta_1$. For any vector field $\mathbf{F}(\mathbf{x})$ defined on Q, one has

$$\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})-\mathbf{F}(\mathbf{x})|^2\rangle \geqslant 0. \tag{2.1}$$

Setting $\mathbf{F}$ equal to a constant vector $\overline{\mathbf{F}}$, one obtains

$$\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2\rangle \geqslant 2\overline{\mathbf{F}}\cdot\langle\chi_2(\mathbf{x})\mathbf{E}(\mathbf{x})\rangle - \theta_2|\overline{\mathbf{F}}|^2. \tag{2.2}$$

Optimizing over $\overline{\mathbf{F}}$ gives

$$\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2\rangle \geqslant \frac{1}{\theta_2}|\langle\chi_2(\mathbf{x})\mathbf{E}(\mathbf{x})\rangle|^2. \tag{2.3}$$

Expanding Eq. (1.4), one obtains

$$\mathcal{E}^e\overline{\mathbf{E}} = \langle[(\epsilon_1+\chi_2(\epsilon_2-\epsilon_1)]\mathbf{E}(\mathbf{x})\rangle. \tag{2.4}$$

Recalling that $\langle\mathbf{E}(\mathbf{x})\rangle=\overline{\mathbf{E}}$, one easily deduces the identity given by

$$(\epsilon_2-\epsilon_1)^{-1}(\mathcal{E}^e-\epsilon_1 I)\overline{\mathbf{E}} = \langle\chi_2(\mathbf{x})\mathbf{E}(\mathbf{x})\rangle. \tag{2.5}$$

From Eq. (2.3), one obtains

$$\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2\rangle \geqslant \frac{1}{\theta_2(\epsilon_2-\epsilon_1)^2}|(\mathcal{E}^e-\epsilon_1 I)\overline{\mathbf{E}}|^2. \tag{2.6}$$

For p and q such that $p\geqslant 1$ and $1/p+1/q=1$, an elementary estimate gives

$$|Q|^{1/q}\theta_2^{1/q}\left(\int_Q \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^{2p}d\mathbf{x}\right)^{1/p} \geqslant |Q|\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2\rangle, \tag{2.7}$$

and it follows that

$$\langle\chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} \geqslant \frac{\theta_2^{1/r}}{\theta_2|\epsilon_2-\epsilon_1|}|(\mathcal{E}^e-\epsilon_1 I)\overline{\mathbf{E}}|, \tag{2.8}$$

for $2\leqslant r\leqslant\infty$. From Eq. (2.5), one easily sees that the lower bound given by Eq. (2.8) is optimal when the electric field is constant inside material two.

Similar arguments give the lower bound

$$\langle\chi_1(\mathbf{x})|\mathbf{E}(\mathbf{x})|^2\rangle \geqslant \frac{1}{\theta_1(\epsilon_1-\epsilon_2)^2}|(\mathcal{E}^e-\epsilon_2 I)\overline{\mathbf{E}}|^2, \tag{2.9}$$

and it follows that

$$\langle\chi_1(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} \geqslant \frac{\theta_1^{1/r}}{\theta_1|\epsilon_1-\epsilon_2|}|(\mathcal{E}^e-\epsilon_2 I)\overline{\mathbf{E}}| \tag{2.10}$$

for $2\leqslant r\leqslant\infty$. Here, equality holds in Eq. (2.10) when the electric-field is constant inside phase one.

III. OPTIMAL LOWER BOUNDS ON THE MOMENTS OF THE ELECTRIC-FIELD

Optimal lower bounds on the moments and L^∞ norms of the electric-field are presented. The bounds are given in terms of the volume fraction of material one and the eigenvalues of a tensor of geometric parameters that depend explicitly on the two-point correlation function. The tensor of geometric parameters $\mathcal{M}(S_1^1,S_1^2)$ is now well known and is given by[2,21]

$$\mathcal{M}(S_1^1,S_1^2) = \frac{1}{S_1^1(1-S_1^1)}\sum_{\mathbf{k}\neq 0} S_1^2(\mathbf{k})\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}, \tag{3.1}$$

where $\mathbf{k}$ is a vector on the integer lattice, $\mathbf{k}\otimes\mathbf{k}$ is the rank-1 matrix with entries $\mathbf{k}_i\mathbf{k}_j$ and $S_1^2(\mathbf{k})$ are the Fourier coefficients of $S_1^2(\mathbf{t})$ computed over the cube Q. Here, $trace\{\mathcal{M}(S_1^1,S_1^2)\}=1$ and $\mathcal{M}(S_1^1,S_1^2)$ is positive semidefinite. The eigenvalues of $\mathcal{M}(S_1^1,S_1^2)$ are written in an ascending order and are denoted by $\lambda_1(S_1^1,S_1^2)$, $\lambda_2(S_1^1,S_1^2)$, and $\lambda_3(S_1^1,S_1^2)$. It is evident that one can introduce the one- and two-point correlation functions for material two denoted by S_2^1 and $S_2^2(\mathbf{t})$. For future reference, we point out that it is well known and easy to see that the associated tensor of geometric parameters $\mathcal{M}(S_2^1,S_2^2)$ is identical to $\mathcal{M}(S_1^1,S_1^2)$.

We introduce the set $\mathcal{K}^+$ of all vectors $\mathbf{d}=(d_1,d_2,d_3)$ such that $0\leqslant d_1\leqslant d_2\leqslant d_3$ and $\Sigma_i\, d_i=1$. The class of microgeometries (configurations of the two materials) for which $S_1^1=\theta_1$ and $\lambda_i(S_1^1,S_1^2)=d_i$, $i=1,2,3$ is denoted by $\mathcal{R}(\theta_1,\mathbf{d})$. In what follows, we provide optimal lower bounds on the moments and L^∞ norms of the electric-field for microstructures in the class $\mathcal{R}(\theta_1,\mathbf{d})$. From a mathematical perspective, this problem is an optimization problem, i.e., among all configurations in $\mathcal{R}(\theta_1,\mathbf{d})$, we seek a configuration of the two dielectrics that minimize the moments and the L^∞ norms. It is shown here that the extremal microgeometries that attain the bounds are given by the confocal-ellipsoid assemblages.

The construction of a confocal-ellipsoid assemblage with a core of material two and a coating of material one is described as follows. One considers the cube containing a space-filling assemblage of ellipsoids. Here, all ellipsoids are contained inside Q and have the same shape and orientation of axes and differ only in their size. Inside each ellipsoid, one places a smaller confocal-ellipsoid filled with material two and the surrounding shell is filled with material one. We call these coated ellipsoids. The part of Q not covered by the coated ellipsoids has zero volume (measure). The volume fractions of materials one and two are the same for each coated ellipsoid and are given by the proportions θ_1 and θ_2, respectively (see Fig. 1.) A confocal-ellipsoid assemblage with a core of material one and a coating of material two is constructed in an identical way.

For future reference, we list the well-known properties[2,3] of the local electric-field inside the confocal-ellipsoid assemblage that are useful for the subsequent analysis. To fix ideas, we consider an assemblage with a core of material two and a coating of material one. We select a prototypical coated ellipsoid from the assemblage. One recalls that the electric field in the composite is given by $\mathbf{E}(\mathbf{x})=\nabla\phi$. Here, ϕ is continuous inside the coated ellipsoid, harmonic in the core phase and coating phase, and satisfies the transmission conditions [Eq. (1.3)] on the core-coating interface. The fields inside the coated ellipsoid exhibit several distinguishing features. The first and foremost is that $\phi=\overline{\mathbf{E}}\cdot\mathbf{x}$ on the external boundary of the coated ellipsoid. This implies that on the external boundary

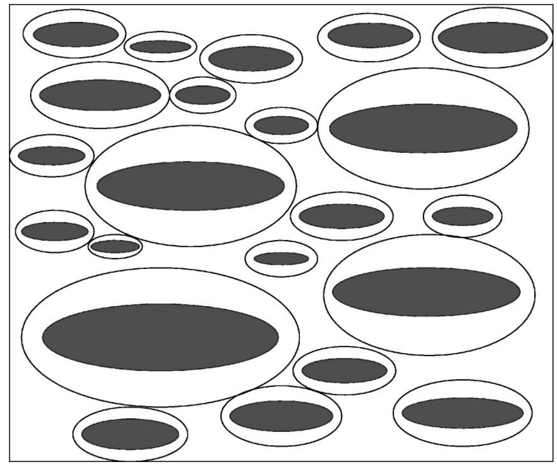

FIG. 1. Confocal-ellipsoid assemblage.

$$\tau\cdot\mathbf{E}(\mathbf{x}) = \tau\cdot\overline{\mathbf{E}} \tag{3.2}$$

for every vector τ tangent to the external boundary at $\mathbf{x}$. Secondly, on the external boundary, one has the following flux condition given by

$$\mathbf{n}\cdot\epsilon_1\mathbf{E}(\mathbf{x}) = \mathbf{n}\cdot\overline{\mathcal{E}^e}\,\overline{\mathbf{E}}, \tag{3.3}$$

where $\mathbf{n}$ is the exterior unit normal and $\overline{\mathcal{E}^e}$ is the effective dielectric constant of the confocal-ellipsoid assemblage. Lastly, the electric-field inside the core material two is constant and given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_2(\epsilon_2-\epsilon_1)}(\overline{\mathcal{E}^e}-\epsilon_1 I)\overline{\mathbf{E}}. \tag{3.4}$$

The confocal-ellipsoid assemblage consists of translated and rescaled versions of the prototypical coated ellipsoid. The electric-field $\tilde{\mathbf{E}}(\mathbf{x})$ in a rescaled and translated coated ellipsoid with scale factor $t>0$ is related to the electric-field $\mathbf{E}(\mathbf{x})$ in the prototype by $\tilde{\mathbf{E}}(\mathbf{x})=\mathbf{E}(t^{-1}\mathbf{x})$ and Eqs. (3.2)–(3.4) are satisfied for every rescaled and translated confocal-ellipsoid. Thus, the electric-field in material two is given by Eq. (3.4) and the lower bound in Eq. (2.8) is attained. Interchanging core and coating materials, one sees that the field inside phase one is constant for a confocal-ellipsoid assemblage with a core of material one and a coating of material two and is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_1(\epsilon_1-\epsilon_2)}(\underline{\mathcal{E}}^e-\epsilon_2 I)\overline{\mathbf{E}}. \tag{3.5}$$

and the lower bound in Eq. (2.10) is attained.

For $\mathbf{d}$ and θ_1 fixed, we define

$$L^1 = \frac{\epsilon_2}{\epsilon_2+(\epsilon_1-\epsilon_2)(1-\theta_1)d_3} \tag{3.6}$$

and

$$L^2 = \frac{\epsilon_1}{\epsilon_1-(\epsilon_1-\epsilon_2)\theta_1 d_1}, \tag{3.7}$$

where $L^1\leqslant 1\leqslant L^2$.

The optimal lower bounds on the moments of the electric-field are given in the following:

A. Optimal lower bound on the moments of the electric-field intensity in material one

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric-field $\mathbf{E}(\mathbf{x})$ associated with any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$ satisfies

$$\theta_1^{1/r}L^1|\overline{\mathbf{E}}| \leq \langle \chi_1|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r}, \quad \text{for} \quad 2 \leq r < \infty. \tag{3.8}$$

Moreover, for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material one in $\mathcal{R}(\theta_1,\mathbf{d})$, for which the minor axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$, and Eq. (3.8) holds with equality for every r in $2 \leq r < \infty$.

B. Optimal lower bound on the L^∞ norms of the electric-field intensity in material one

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric-field $\mathbf{E}(\mathbf{x})$ associated with any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$ satisfies

$$L^1|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_1)}. \tag{3.9}$$

Moreover, for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material one in $\mathcal{R}(\theta_1,\mathbf{d})$, for which the minor axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$, and Eq. (3.9) holds with equality.

C. Optimal lower bound on the moments of the electric-field intensity in material two

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric-field $\mathbf{E}(\mathbf{x})$ associated with any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$ satisfies

$$(1-\theta_1)^{1/r}L^2|\overline{\mathbf{E}}| \leq \langle \chi_2|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r}, \quad \text{for} \quad 2 \leq r < \infty. \tag{3.10}$$

Moreover, for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material two in $\mathcal{R}(\theta_1,\mathbf{d})$, for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$, and Eq. (3.10) holds with equality for every r in $2 \leq r < \infty$.

D. Optimal lower bound on the L^∞ norm of the electric-field intensity in material two

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric-field $\mathbf{E}(\mathbf{x})$ associated with any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$ satisfies

$$L^2|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)}. \tag{3.11}$$

Moreover, for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material two in $\mathcal{R}(\theta_1,\mathbf{d})$, for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$, and Eq. (3.11) holds with equality.

E. Optimal lower bound on the L^∞ norm of the electric-field intensity

For every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, the electric-field $\mathbf{E}(\mathbf{x})$ associated with any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$ satisfies

$$L^2|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{3.12}$$

Moreover, for every θ_1 and $\mathbf{d}$ in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material two in $\mathcal{R}(\theta_1,\mathbf{d})$, for which the major axis of the ellipsoids are aligned with $\overline{\mathbf{E}}$, and Eq. (3.12) holds with equality.

When the composite is statistically isotropic, $\mathbf{d}=(1/3,1/3,1/3)$ and

$$L^2 = 3\epsilon_1/[3\epsilon_1-(\epsilon_1-\epsilon_2)\theta_1]. \tag{3.13}$$

For this case, one has the following optimal lower bound on the L^∞ norm of the electric-field intensity inside the composite given by the following:

F. Optimal lower bound on the L^∞ norm of the electric-field intensity for statistically isotropic composites

Consider all microgeometries in $\mathcal{R}(\theta_1,\mathbf{d})$ for the case $\mathbf{d}=(1/3,1/3,1/3)$. For a prescribed average electric-field $\overline{\mathbf{E}}$, the lower bound on the L^∞ norm of the electric-field concentration is given by

$$\{3\epsilon_1/[3\epsilon_1-(\epsilon_1-\epsilon_2)\theta_1]\}|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{3.14}$$

The lower bound is attained by the electric-field inside the Hashin-Shtrikman concentric-coated sphere assemblage with a core of material two for every choice of applied field $\overline{\mathbf{E}}$.

IV. DERIVATION OF THE LOWER BOUNDS

We recall the energy bounds[21] given by

$$\mathcal{U}^-\eta\cdot\eta \leq \mathcal{E}^e\eta\cdot\eta \leq \mathcal{U}^+\eta\cdot\eta \tag{4.1}$$

for every constant vector η, Where

$$\mathcal{U}^+ = \epsilon_1 I - (1-S_1^1)\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1}I - S_1^1\mathcal{M}(S_1^1,S_1^2)]^{-1}, \tag{4.2}$$

$$\mathcal{U}^- = \epsilon_2 I + S_1^1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1}I + (1-S_1^1)\mathcal{M}(S_1^1,S_1^2)]^{-1}. \tag{4.3}$$

We write the eigenvalues of $\mathcal{E}^e$ in an ascending order ϵ_1^e, ϵ_2^e, and ϵ_3^e. For any microgeometry in $\mathcal{R}(\theta_1,\mathbf{d})$, it follows easily from Eq. (4.1) that

$$\underline{\lambda}_i \leq \epsilon_i^e \leq \overline{\lambda_i}, \quad \text{for} \quad i=1,2,3, \tag{4.4}$$

where

$$\begin{aligned}
\underline{\lambda}_1 &= \epsilon_2 + \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_3]^{-1},\\
\overline{\lambda}_1 &= \epsilon_1 - \theta_2\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_3]^{-1};\\
\underline{\lambda}_2 &= \epsilon_2 + \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_2]^{-1};\\
\overline{\lambda}_2 &= \epsilon_1 - \theta_2\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_2]^{-1};\\
\underline{\lambda}_3 &= \epsilon_2 + \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1} + \theta_2 d_1]^{-1},\\
\overline{\lambda}_3 &= \epsilon_1 - \theta_2\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1} - \theta_1 d_1]^{-1}.
\end{aligned} \tag{4.5}$$

It follows immediately from Eqs. (4.4) and (4.5) that

$$L^1|\overline{\mathbf{E}}| \leq \frac{1}{\theta_1|\epsilon_1-\epsilon_2|}|(\mathcal{E}^e-\epsilon_2 I)\overline{\mathbf{E}}|,$$
$$L^2|\overline{\mathbf{E}}| \leq \frac{1}{\theta_2|\epsilon_1-\epsilon_2|}|(\mathcal{E}^e-\epsilon_1 I)\overline{\mathbf{E}}|, \tag{4.6}$$

and the lower bounds in Eqs. (3.9) and (3.11) now follow immediately from Eqs. (2.8), (2.10), and (4.6).

To obtain bounds on the moments, note that Eqs. (2.6), (2.9), and (4.6) together with Hölders' inequality imply that

$$|Q|\theta_1(L^1)^2|\overline{\mathbf{E}}|^2 \leq \int_Q \chi_1|\mathbf{E}(\mathbf{x})|^2\,d\mathbf{x} \leq (|Q|\theta_1)^{1/q}\left(\int_Q \chi_1|\mathbf{E}(\mathbf{x})|^{2p}\,d\mathbf{x}\right)^{1/p},$$

$$|Q|\theta_2(L^2)^2|\overline{\mathbf{E}}|^2 \leq \int_Q \chi_2|\mathbf{E}(\mathbf{x})|^2\,d\mathbf{x} \leq (|Q|\theta_2)^{1/q}\left(\int_Q \chi_2|\mathbf{E}(\mathbf{x})|^{2p}\,d\mathbf{x}\right)^{1/p}, \tag{4.7}$$

where $p\geq 1$, $1/p+1/q=1$, and the lower bounds in Eqs. (3.8) and (3.10) follow immediately.

The lower bound given by Eq. (3.12) follows immediately from Eq. (3.11) and

$$L^2|\overline{\mathbf{E}}| \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q_2)} \leq \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)}. \tag{4.8}$$

V. OPTIMALITY

In this section, the lower bounds are shown to be attained within the class of microstructures given by the confocal-ellipsoid assemblages. We write the formulas for the effective dielectric tensors for the confocal-ellipsoid assemblage in a form that is suitable for our purposes. The formulas differ from the original formulas given in Refs. 2 and 3 and are derived in the following sections. The effective dielectric tensor for the confocal-ellipsoid assemblage with a core of material two and a coating of material one is given by

$$\epsilon_1 I - \overline{\mathcal{E}^e} = \theta_2\epsilon_1[\epsilon_1(\epsilon_1-\epsilon_2)^{-1}I - \theta_1\mathcal{M}(S_1^1,S_1^2)]^{-1}, \tag{5.1}$$

and the eigenvalues are given by $\overline{\lambda}_i$, for $i=1$, 2, and 3 defined in Eq. (4.5). Eigenvectors corresponding to $\overline{\lambda}_3$ include vectors parallel to the major axis of the coated ellipsoids. Eigenvectors corresponding to $\overline{\lambda}_1$ include vectors parallel to the minor axis of the coated ellipsoids.

The effective dielectric tensor with a core of material one and a coating of material two is given by

$$\underline{\mathcal{E}}^e - \epsilon_2 I = \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1}I + \theta_2\mathcal{M}(S_1^1,S_1^2)]^{-1}, \tag{5.2}$$

and the eigenvalues are given by $\underline{\lambda}_i$, for $i=1$, 2, and 3 defined in Eq. (4.5). Here, eigenvectors corresponding to $\underline{\lambda}_3$ include vectors parallel to the major axis of the coated ellipsoids. Eigenvectors corresponding to $\underline{\lambda}_1$ include vectors parallel to the minor axis of the coated ellipsoids.

We now state the following:

A. Attainability property

For any choice of θ_1 and any symmetric tensor $\mathcal{H}$ with eigenvalues in $\mathcal{K}^+$, there exists a confocal-ellipsoid assemblage with a core of material one and a coating of material two, such that $S_1^1=\theta_1$,

$$\mathcal{M}(S_1^1,S_1^2) = \mathcal{H}, \tag{5.3}$$

and there exists a confocal-ellipsoid assemblage with a core of material two and a coating of material one, such that $S_1^1=\theta_1$ and

$$\mathcal{M}(S_2^1,S_2^2) = \mathcal{H}. \tag{5.4}$$

The attainability property is established in the following section.

We now establish the optimality of the lower bound in Eq. (3.8). The characteristic function of material one in the confocal-ellipsoid assemblage with a core of material one and a coating of material two is denoted by χ_1^{el}. The volume fraction of material one is prescribed to be θ_1, i.e., $\langle\chi_1^{el}\rangle=\theta_1$. The electric-field inside material one of the confocal-ellipsoid assemblage is constant and from Eq. (3.5), one recalls that the electric-field in material one is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_1(\epsilon_1-\epsilon_2)}(\underline{\mathcal{E}}^e-\epsilon_2 I)\overline{\mathbf{E}}. \tag{5.5}$$

For a prescribed vector $\mathbf{d}$ in $\mathcal{K}^+$, it follows from Eq. (5.3) that we can choose the confocal-ellipsoid assemblage, such that the eigenvalues of $\mathcal{M}(S_1^1,S_1^2)$ correspond to $\mathbf{d}$ and

$$\mathcal{M}(S_1^1,S_1^2)\overline{\mathbf{E}} = d_3\overline{\mathbf{E}}. \tag{5.6}$$

For this case, it follows from earlier remarks that the minor axes of the ellipsoids are aligned with the applied field $\overline{\mathbf{E}}$ and from Eqs. (5.2), (5.5), and (5.6), it follows that the electric-field in material one is given by

$$\mathbf{E}(\mathbf{x}) = \frac{1}{\theta_1(\epsilon_1-\epsilon_2)}(\underline{\lambda}_1-\epsilon_2 I)\overline{\mathbf{E}} = L^1\overline{\mathbf{E}}. \tag{5.7}$$

Substitution gives

$$\langle\chi_1^{el}|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} = \theta_1^{1/r}L^1|\overline{\mathbf{E}}|, \tag{5.8}$$

and the optimality of Eq. (3.8) is established. The optimality of Eq. (3.9) follows immediately from Eq. (5.7). The optimality of the bounds in Eqs. (3.10) and (3.11) follow from identical considerations using confocal-ellipsoid assemblages with a core of material two and a coating of material one. For this case, the major axes of the ellipsoids are aligned with $\overline{\mathbf{E}}$ and $\mathbf{E}(\mathbf{x})=L^2\overline{\mathbf{E}}$ in the core.

Last, we establish the optimality of Eq. (3.12). To do this, we examine the electric-field inside the confocal-ellipsoid assemblage with a core of material two for which all ellipsoids have major axes aligned with the applied field $\overline{\mathbf{E}}$. Consider any coated ellipsoid in the assemblage and note that $|\mathbf{E}(\mathbf{x})|=L^2|\overline{\mathbf{E}}|$ in the core. In what follows, we will show that the electric-field intensity in the coating is bounded above by $L^2|\overline{\mathbf{E}}|$. Because the coated ellipsoids cover Q (up to a set of measure zero), it then follows that

$$L^2|\overline{E}| = \|\mathbf{E}(\mathbf{x})\|_{L^\infty(Q)} \tag{5.9}$$

for this confocal-ellipsoid assemblage.

Now we show that the electric-field intensity is bounded above by $L^2|\overline{\mathbf{E}}|$ in the coating. To do this, recall that $|\mathbf{E}(\mathbf{x})| = |\nabla\phi(\mathbf{x})|$ in the coating phase and that ϕ is harmonic there. Thus, $|\mathbf{E}(\mathbf{x})|$ is subharmonic in the coating, and from the maximum principle, it necessarily takes its maximum values either on the interface between the core and the coating or on the external boundary of the coated ellipsoid. We denote tangent vectors to the core–coating interface by τ, and the unit normal to the interface is denoted by $\mathbf{n}$. The trace of the electric-field on the core side of the interface is denoted by $\mathbf{E}_{|2}$, and the trace of the electric-field on the coating side of the interface is denoted by $\mathbf{E}_{|1}$. Continuity of ϕ across the interface gives

$$\mathbf{E}_{|1}\cdot\tau = \mathbf{E}_{|2}\cdot\tau, \tag{5.10}$$

and continuity of the normal component of displacement gives

$$\epsilon_1\mathbf{E}_{|1}\cdot\mathbf{n} = \epsilon_2\mathbf{E}_{|2}\cdot\mathbf{n}. \tag{5.11}$$

For points on the interface where $\mathbf{E}_{|_1}\cdot\mathbf{n}=0$, it is clear that $|\mathbf{E}_{|1}| = |\mathbf{E}_{|2}|$. For all other points on the interface

$$|\mathbf{E}_{|1}\cdot\mathbf{n}| = \frac{\epsilon_2}{\epsilon_1}|\mathbf{E}_{|2}\cdot\mathbf{n}| \leqslant |\mathbf{E}_{|2}\cdot\mathbf{n}|, \tag{5.12}$$

since $\epsilon_1 > \epsilon_2$. It now follows from Eqs. (5.10) and (5.12) that

$$|\mathbf{E}_{|1}| \leqslant |\mathbf{E}_{|2}| = L^2|\overline{\mathbf{E}}| \tag{5.13}$$

on the core–coating interface. The trace of the electric-field on the coating side of the external interface is denoted by $\mathbf{E}_{|ext}$. On the external boundary of the coated ellipsoid, one recalls Eq. (3.2) and

$$\mathbf{E}_{|ext}\cdot\tau = \overline{\mathbf{E}}\cdot\tau, \tag{5.14}$$

where τ is any tangent vector to the external boundary. From Eq. (3.3), we have

$$\epsilon_1\mathbf{E}_{|ext}\cdot\mathbf{n} = \overline{\mathcal{E}^e}\,\overline{\mathbf{E}}\cdot\mathbf{n} = \overline{\lambda_3}\,\overline{\mathbf{E}}\cdot\mathbf{n}. \tag{5.15}$$

Here, $\mathbf{n}$ is the outer unit normal to the external boundary and from Eq. (4.5)

$$\overline{\lambda_3} < \epsilon_1. \tag{5.16}$$

Using, Eq. (5.16), we argue as before to deduce that on the outer boundary $|\mathbf{E}_{|ext}| \leqslant |\overline{\mathbf{E}}|$. We apply the maximum principle and note that $1 \leqslant L^2$ to deduce that $|\mathbf{E}(\mathbf{x})| \leqslant L^2|\overline{\mathbf{E}}|$ in the coating and optimality follows.

Optimality of Eq. (3.14) follows immediately from the same arguments used to establish the optimality of Eq. (3.12). This can also be checked by directly calculating the electric field inside the Hashin-Shtrikman coated sphere assemblage.

VI. FORMULAS FOR EFFECTIVE PROPERTIES

In this section, we establish the formulas in Eqs. (5.1) and (5.2) for the effective properties of confocal-ellipsoid assemblages. We equate these formulas with the better-known formulas for the effective properties of confocal-ellipsoid assemblages given in Refs. 2 and 3 to establish the attainability property expressed in Eqs. (5.3) and (5.4).

We sketch the ideas behind the derivation of Eq. (5.2) noting that Eq. (5.1) is established along similar lines. The electric-field inside the concentric-ellipsoid assemblage with a core of material one and a coating of material two admits the decomposition $\mathbf{E}(\mathbf{x}) = \nabla u + \overline{\mathbf{E}}$, where u is Q periodic and is the solution of

$$\mathrm{div}\{[\epsilon_1\chi_1 + \epsilon_2(1-\chi_1)](\nabla u + \overline{\mathbf{E}})\} = 0. \tag{6.1}$$

This is equivalent to

$$\epsilon_2\Delta u = -\,\mathrm{div}[(\epsilon_1-\epsilon_2)\chi_1(\nabla u + \overline{\mathbf{E}})] = 0. \tag{6.2}$$

From Eq. (6.2), one easily obtains the identity

$$\mathbf{E}(\mathbf{x}) = -\frac{(\epsilon_1-\epsilon_2)}{\epsilon_2}\nabla\,\Delta^{-1}\mathrm{div}(\chi_1\mathbf{E}(\mathbf{x})) + \overline{\mathbf{E}}, \tag{6.3}$$

where $w = \Delta^{-1}f$ is the Q periodic solution of $\Delta w = f$. The constant value that $\mathbf{E}(\mathbf{x})$ takes in material one is denoted by $\tilde{\mathbf{E}}$ and it is clear that $\chi_1(\mathbf{x})\mathbf{E}(\mathbf{x}) = \chi_1(\mathbf{x})\tilde{\mathbf{E}}$. Multiplying Eq. (6.3) by χ_1 and taking averages gives

$$\theta_1\tilde{\mathbf{E}} = -\frac{(\epsilon_1-\epsilon_2)}{\epsilon_2}\langle\chi_1\nabla\,\Delta^{-1}\mathrm{div}\;\chi_1\tilde{\mathbf{E}}\rangle + \theta_1\overline{\mathbf{E}}. \tag{6.4}$$

It is well known[2] that

$$\langle\chi_1\nabla\,\Delta^{-1}\;\mathrm{div}\;\chi_1\tilde{\mathbf{E}}\rangle = \theta_1\theta_2\mathcal{M}(S_1^1,S_1^2)\tilde{\mathbf{E}}, \tag{6.5}$$

and it follows that

$$\tilde{\mathbf{E}} = \theta_1\left(\theta_1 I + \frac{(\epsilon_1-\epsilon_2)}{\epsilon_2}\theta_1\theta_2\mathcal{M}(S_1^1,S_1^2)\right)^{-1}\overline{\mathbf{E}}. \tag{6.6}$$

We recall from Eq. (3.5) that

$$\tilde{\mathbf{E}} = \frac{1}{\theta_1(\epsilon_1-\epsilon_2)}(\underline{\mathcal{E}}^e - \epsilon_2 I)\overline{\mathbf{E}}. \tag{6.7}$$

Equation (5.2) follows on elimination of $\tilde{\mathbf{E}}$ from Eqs. (6.6) and (6.7).

We establish the attainability result stated in Eq. (5.3). From Refs. 2 and 3, the effective dielectric tensor with a core of material one and a coating of material two is given by

$$\underline{\mathcal{E}}^e - \epsilon_2 I = \theta_1\epsilon_2[\epsilon_2(\epsilon_1-\epsilon_2)^{-1}I + \theta_2\mathcal{H}]^{-1}, \tag{6.8}$$

where $\mathcal{H}$ is a symmetric-positive-semidefinite matrix with a unit trace. It is shown in Ref. 3 that $\mathcal{H}$ ranges over all such matrices as the shape of the ellipsoids are varied while keeping the core volume fraction θ_1 fixed. The attainability property in Eq. (5.3) follows immediately noting that Eqs. (5.2) and (6.8) are equal and solving for $\mathcal{M}(S_1^1,S_1^2)$. Identical arguments are used to establish Eq. (5.4).

VII. BOUNDS ON FIELD CONCENTRATIONS FOR TWO-PHASE COMPOSITES WITH COMPLEX DIELECTRIC PERMITTIVITY

The effects of loss become important when considering optical properties of composite materials. For this case, the dielectric constants ϵ_1 and ϵ_2 are complex. A straightforward calculation easily shows that the lower bounds on the field fluctuations given by Eqs. (2.8) and (2.10) also hold for this case. Moreover, Eq. (2.8) is optimal when the electric-field is constant in material two, and Eq. (2.10) is optimal when the electric-field is constant in material one.

The methodology of Sec. III is applied to obtain lower bounds on the field concentrations in statistically isotropic two-phase dielectric composites when the dielectric constants are complex. To illustrate the method, we show how to obtain a lower bound on the field concentration inside material two. For isotropic composites, the complex effective dielectric tensor reduces to $\mathcal{E}^e=\epsilon^e I$. Here, ϵ^e is the effective complex permittivity. For this case, Eq. (2.8) becomes

$$\langle \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} \geqslant \theta_2^{1/r}\left(\frac{1}{\theta_2}\left|\frac{\epsilon^e-\epsilon_1}{\epsilon_2-\epsilon_1}\right|\right)|\overline{\mathbf{E}}| \tag{7.1}$$

for $2\leqslant r\leqslant\infty$.

Bounds on ϵ^e that are given in terms of the volume fractions and dielectric constants of the component materials were derived in Ref. 22. These reduce to the Hashin-Shtrikman bounds when the component dielectrics are real valued. In this context, the bounds are given by curves bounding a region Ω_{θ_2} of the complex plane inside, which ϵ^e must lie. The explicit formulas for the boundary are given in terms of ϵ_1, ϵ_2, and θ_2 and can be found in Ref. 2. We set

$$L^2=\min\left\{\frac{1}{\theta_2}\left|\frac{z-\epsilon_1}{\epsilon_2-\epsilon_1}\right|; z \text{ in } \Omega_{\theta_2}\right\}. \tag{7.2}$$

The lower bound on the field concentrations for statistically isotropic composites is given by

$$\theta_2^{1/r}L^2|\overline{\mathbf{E}}| \leqslant \langle \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r} \tag{7.3}$$

for $2\leqslant r\leqslant\infty$. Here, the constant L^2 is computed numerically. To fix ideas for $\epsilon_1=20+i$, $\epsilon_2=-2+3i$, and $\theta_2=0.4$, calculation shows that $L^2=1.06$. Similar considerations give lower bounds on the field fluctuations in material one. It is pointed out that, for the case of complex dielectric constants, the optimality of the bound in Eq. (7.3) remains an open question and is the topic of future research.

If more information on the microstructure is available, then one has tighter lower bounds on the higher moments of the electric-field. This follows from the correspondingly tighter bounds on ϵ^e given in terms of higher-order statistical information.[2] The tighter bounds restrict ϵ^e to a subset $\mathcal{A}$ of Ω_{θ_2}. It is then clear from Eq. (7.1) that minimization of

$$\frac{1}{\theta_2}\left|\frac{z-\epsilon_1}{\epsilon_2-\epsilon_1}\right| \tag{7.4}$$

for z in $\mathcal{A}$ provides a lower bound on $\langle \chi_2(\mathbf{x})|\mathbf{E}(\mathbf{x})|^r\rangle^{1/r}$ that is greater than or equal to $\theta_2^{1/r}L^2|\overline{\mathbf{E}}|$.

VIII. DISCUSSION: EXTREME SUPPRESSION OF FIELD CONCENTRATIONS

One recalls that the L^∞ norms of the electric-field intensity inside the confocal-ellipsoid assemblage with a core of material two and a major axis aligned with $\overline{\mathbf{E}}$ is given by $\epsilon_1[\epsilon_1-(\epsilon_1-\epsilon_2)\theta_1 d_1]^{-1}|\overline{\mathbf{E}}|$. In the limit when the ratio between the major and minor axes of the ellipsoids tends to be infinite the geometric parameter d_1 tends to be zero and the L^∞ norms of the field intensity, is precisely $|\overline{\mathbf{E}}|$. This value agrees with the electric-field intensity seen in a layered material with layers parallel to $\overline{\mathbf{E}}$. The largest value of the L^∞ norm of the electric-field intensity for this class of assemblages with a core of material two and major axes aligned with $\overline{\mathbf{E}}$ is given by the Hashin-Shtrikman coated sphere assemblage when all axes of the ellipsoids are equal and $d_1=1/3$.

It is clear that the L^∞ norm of the electric-field intensity inside the confocal-ellipsoid assemblage with a core of material two remains finite even as $\epsilon_2\to 0$. In this limit, it is given by $1/1-\theta_1 d_1|\overline{\mathbf{E}}|$. At first sight, this appears counterintuitive as high-contrast inclusions can be arbitrarily close together inside the confocal-ellipsoid assemblage. However, since every coated ellipsoid "sees" the linear Dirichlet boundary conditions given by $\phi=\overline{\mathbf{E}}\cdot\mathbf{x}$, it is clear that the fields inside each coated ellipsoid are not affected by the surrounding inclusions, and field concentrations do not occur.

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